HOLOMORPHIC EXTENSION ASSOCIATED WITH FOURIER–LEGENDRE EXPANSIONS

E. DE MICHELI AND G. A. VIANO

Abstract. In this article we prove that if the coefficients of a Fourier–Legendre expansion satisfy a suitable Hausdorff–type condition, then the series converges to a function which admits a holomorphic extension to a cut–plane. Furthermore, we prove that a Laplace–type (Laplace composed with Radon) transform of the function describing the jump across the cut is the unique Carlsonian interpolation of the Fourier coefficients of the expansion. We can thus reconstruct the discontinuity function from the coefficients of the Fourier–Legendre series by the use of the Pollaczek polynomials.

1. Introduction

Let us consider the following Fourier–Legendre series

$$\frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)a_n P_n(\cos\theta). \tag{1.1}$$

The classical theory of polynomial expansions, as described by Walsh [10], establishes for these expansions convergence properties which are closely analogous to the well–known convergence properties of the Taylor series expansions. In this case the region of convergence, instead of being circles of radius ρ, are ellipses E_ρ with foci ± 1 and *radius* ρ =(semiminor+semimajor)–axis. The expansion converges inside the largest ellipse within which the function being expanded in terms of the series (1.1) is holomorphic.

The following question quite naturally arises: Is it possible to find suitable conditions on the coefficients a_n which allow a holomorphic extension of the function (to which the series (1.1) converges) to the whole complex $\cos\theta$–plane except for a cut along the positive axis? The answer to this question is positive and quite analogous to that derived in [5] in connection with the Taylor and Laurent series: essentially, the coefficients $\{a_n\}$ are required to satisfy suitable Hausdorff–type conditions.

To prove these results it is convenient to proceed through two steps. First, replacing the complex $\cos\theta$–plane ($\theta\in\mathbb{C}$; $\theta=u+iv$; $u,v\in\mathbb{R}$) by a complex hyperboloid $X^{(c)}$, which contains as submanifolds the *Euclidean sphere* $S=(i\mathbb{R}\times\mathbb{R}^2)\cap X^{(c)}$ which gives the support of the SO(3,$\mathbb{R}$) harmonic analysis, and the real one–sheeted hyperboloid $X=\mathbb{R}^3\cap X^{(c)}$ that contains the support of the cut (see Fig. 1). In the second step we consider a fibration on a meridian hyperbola of $X^{(c)}$, which is obtained through a Radon–type transformation. This fibration allows us to reduce the harmonic analysis to that associated with a complex one–dimensional hyperbola, which *contains* the Euclidean–circle and the real hyperbola. We are thus led to regard the series (1.1) as a trigonometrical series on the Euclidean–circle, making

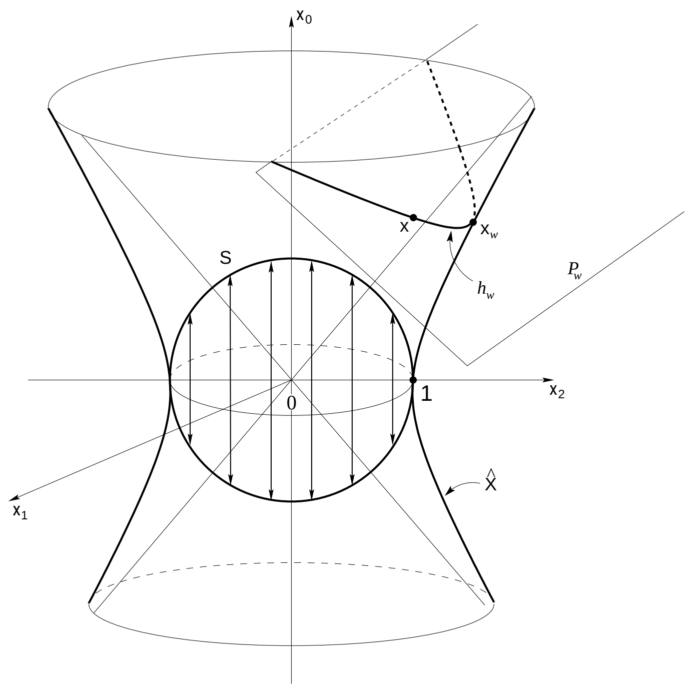

Figure 1: Horocyclic fibration of the one–sheeted hyperboloid.

it possible to apply several results of the same type as those obtained in [5]. In particular, we can prove that this trigonometrical series converges to a function which admits a holomorphic extension to a complex cut–plane if the coefficients a_n satisfy a suitable Hausdorff–type condition. Then, by inverting the Radon transform, we return to the complex θ–plane and, finally, study the holomorphic extension associated with the Legendre series (1.1). In addition, we obtain a unique Carlsonian interpolation of the a_n's, denoted by $\tilde{a}(\lambda)$, which turns out to be the composition of the ordinary Laplace transform with the Radon transform. By inverting these Laplace and Radon transforms, we are then able to reconstruct the jump function across the cut using the Pollaczek polynomials.

In a classical article [8], Stein and Wainger have already proved the holomorphic extension associated with the series (1.1) (see, in particular, Theorem 4 in their article). However, our work differs from Stein and Wainger's for several reasons:

i) We associate the holomorphic extension of the Legendre series to the Hausdorff condition on the coefficients a_n, and, accordingly, we introduce the Carlsonian interpolation $\tilde{a}(\lambda)$ of these coefficients which is the composition of an ordinary Laplace transform with a Radon transform (called *spherical–Laplace transform* in [6]).

ii) The inversion of this *Laplace composed with Radon* transform allows us to reconstruct the jump function across the cut starting from the coefficients $\{a_n\}$ of the series (1.1). This procedure is extremely relevant in the inverse problem in quantum scattering theory for the class of Yukawian potentials

(see [7]). In fact, from the discontinuity function across the cut the spectral density associated with the Yukawian potentials can be determined.

iii) For particular values of λ, i.e., $\lambda = -1/2 + i\nu$ ($\nu \in \mathbb{R}$), we obtain from $\tilde{a}(\lambda)$ the classical Mehler transform, which is precisely the mathematical tool used by Stein and Wainger for obtaining their results, through the Plancherel theorem.

iv) Finally, we can give a geometrical interpretation of our results and methods by introducing the complex hyperboloid $X^{(c)}$ and associating to it, through a Radon transform, a fibration on a meridian hyperbola.

This paper is organized as follows: in Section 2 we study the Carlsonian interpolation of the Hausdorff moments and, correspondingly, the Hardy spaces to which this interpolation belongs. In Section 3 we prove that the Legendre expansion can be regarded as a trigonometrical series. In Section 4 we prove a holomorphic extension associated with these trigonometrical series in the complex τ–plane using the same procedure adopted in [5]. The variable τ can then be interpreted as one of the horocyclic coordinates related to the fibration on the meridian hyperbola $\hat{X}^{(c)}$ (see the Appendix). In Section 5 we study the inversion of the Radon–Abel transformation and we prove the holomorphic extension in the θ–plane associated with the Legendre series. In Section 6 we show that the Carlsonian interpolation $\tilde{a}(\lambda)$ of the Hausdorff moments can be represented as the composition of the ordinary Laplace transform with the Radon transform, and we find an integral representation of the jump function across the cut; moreover, we show how to reconstruct the discontinuity function (across the cut) starting from the Fourier–Legendre coefficients $\{a_n\}$, using the Pollaczek polynomials. Finally, in the Appendix we illustrate all the geometrical aspects of the method we used.

2. Interpolation of Hausdorff moments and Hardy spaces

Let us consider a sequence $\{f_n\}_0^\infty$ of (real) numbers f_n, and denote by Δ the difference operator

$$\Delta f_n = f_{n+1} - f_n. \tag{2.1}$$

Then we have:

$$\Delta^k f_n = \underbrace{\Delta \times \Delta \times \cdots \times \Delta}_{k} f_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} f_{n+k-m}, \tag{2.2}$$

(for every $k \geq 0$); Δ^0 is the identity operator by definition. Now, suppose that there exists a positive constant M such that:

$$(n+1)^{(1+\epsilon)} \sum_{i=0}^{n} \binom{n}{i}^{(2+\epsilon)} \left|\Delta^i f_{(n-i)}\right|^{(2+\epsilon)} < M \qquad (n = 0, 1, 2, \ldots; \epsilon > 0). \tag{2.3}$$

It can be proved [11] that condition (2.3) is necessary and sufficient to represent the sequence $\{f_n\}_0^\infty$ as follows:

$$f_n = \int_0^1 x^n u(x)\, dx \qquad (n = 0, 1, 2, \ldots), \tag{2.4}$$

where $u(x)$ belongs to $L^{2+\epsilon}[0,1]$.

We can prove the following proposition.

Proposition 1. *Let the sequence* $\{f_n\}_0^\infty$, $f_n = n^p a_n$, $(p \geq 1)$, *satisfy condition (2.3). Then there exists a unique Carlsonian interpolation of the numbers* a_n, *denoted by* $\tilde{a}(\lambda)$ $(\lambda \in \mathbb{C}, [\tilde{a}(\lambda)]_{(\lambda=n)} = a_n, n = 0,1,2,\ldots)$, *that satisfies the following conditions:*

i) $\tilde{a}(\lambda)$ *is holomorphic in the half–plane* $\mathrm{Re}\,\lambda > -1/2$, *continuous at* $\mathrm{Re}\,\lambda = -1/2$;

ii) $\lambda^p \tilde{a}(\lambda)$ *belongs to* $L^2(-\infty,+\infty)$ *for any fixed value of* $\mathrm{Re}\,\lambda \geq -1/2$: *i.e., putting* $\lambda = \sigma + i\nu$,

$$\int_{-\infty}^{+\infty} \left|(\sigma+i\nu)^p\,\tilde{a}(\sigma+i\nu)\right|^2\,d\nu < \infty; \tag{2.5}$$

iii) $\lambda^p \tilde{a}(\lambda)$ *tends uniformly to zero as* λ *tends to infinity inside any fixed half–plane* $\mathrm{Re}\,\lambda \geq \delta > -1/2$;

iv) $\lambda^{(p-1)}\tilde{a}(\lambda)$ *belongs to* $L^1(-\infty,+\infty)$ *for any fixed value of* $\mathrm{Re}\,\lambda \geq -1/2$.

Proof. If the sequence $\{f_n\}_0^\infty$ satisfies condition (2.3), then representation (2.4) holds true. If we put $x = e^{-t}$ in the integral of (2.4) we obtain:

$$f_n = \int_0^{+\infty} e^{-nt}e^{-t}u(e^{-t})\,dt \qquad (n = 0,1,2,\ldots). \tag{2.6}$$

Therefore the numbers f_n can be regarded as the restriction to the integers of the following Laplace transform:

$$\tilde{F}(\lambda) = \int_0^{+\infty} e^{-(\lambda+1/2)t}e^{-t/2}u(e^{-t})\,dt. \tag{2.7}$$

It can easily be verified that $[\tilde{F}(\lambda)]_{(\lambda=n)} = f_n$. By applying the Paley–Wiener theorem to equality (2.7), and recalling that the function $\exp(-t/2)u(\exp(-t))$ belongs to $L^2[0,+\infty)$, we can conclude that $\tilde{F}(\lambda)$ belongs to the Hardy space $\mathbb{H}^2(\mathbb{C}_{-1/2})$ $(\mathbb{C}_{-1/2} = \{\lambda \in \mathbb{C}, \mathrm{Re}\,\lambda > -1/2\})$. We can thus apply the Carlson theorem [3], and state that $\tilde{F}(\lambda)$ is the unique Carlsonian interpolation of the numbers f_n. Furthermore, by noting that $\tilde{F}(\lambda) = \lambda^p\tilde{a}(\lambda)$ $(p \geq 1)$, properties (ii), (iii) and the analyticity of $\tilde{a}(\lambda)$ in the half–plane $\mathrm{Re}\,\lambda > -1/2$ follow. Next let us note that the function $\exp(-t/2)u(\exp(-t))$ belongs to $L^1[0,+\infty)$; in fact, we can state that $\int_0^{+\infty}|\exp(-t/2)u(\exp(-t))|\,dt = \int_0^1 |u(x)/\sqrt{x}|\,dx < \infty$, in view of the fact that $u \in L^{2+\epsilon}[0,1]$. Therefore, from the Riemann–Lebesgue theorem applied to representation (2.7) it follows that the function $\tilde{F}(-1/2+i\nu)$ $(\nu \in \mathbb{R})$ is continuous, and thus property (i) is proved.

Concerning property (iv) we may use the Schwarz inequality and write

$$\begin{aligned}\int_{-\infty}^{+\infty}\left|(\sigma+i\nu)^{(p-1)}\tilde{a}(\sigma+i\nu)\right|\,d\nu &= \int_{-\infty}^{+\infty}\left|\frac{\tilde{F}(\sigma+i\nu)}{(\sigma+i\nu)}\right|\,d\nu \\ &\leq \left(\int_{-\infty}^{+\infty}\frac{1}{|(\sigma+i\nu)|^2}\,d\nu\right)^{1/2}\left(\int_{-\infty}^{+\infty}|\tilde{F}(\sigma+i\nu)|^2\,d\nu\right)^{1/2} < \infty,\end{aligned} \tag{2.8}$$

if $\sigma \geq -1/2$, $\sigma \neq 0$, $p \geq 1$. In fact, let us note that $\tilde{F}(\sigma+i\nu) \in L^2(-\infty,+\infty)$ for any fixed value of $\mathrm{Re}\,\lambda = \sigma \geq -1/2$. Finally, in view of the regularity and integrability of the function $\lambda^{(p-1)}\tilde{a}(\lambda)$ in the neighborhood of $\mathrm{Re}\,\lambda = 0$, we can conclude that $\lambda^{(p-1)}\tilde{a}(\lambda)$ belongs to $L^1(-\infty,+\infty)$ for any fixed value of $\mathrm{Re}\,\lambda = \sigma \geq -1/2$, $(p \geq 1)$. □

Remark. In order to prove the continuity of $\tilde{a}(\lambda)$ at $\mathrm{Re}\,\lambda = -1/2$ ($\lambda = -1/2+i\nu$, $\nu \in \mathbb{R}$), it is necessary to use condition (2.3) which is slightly more restrictive than condition (8) of [5] where the term ϵ was missing.

3. Legendre expansions as trigonometrical series

Let us consider the following Legendre series:

$$\frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)\,a_n\,P_n(\cos u), \tag{3.1}$$

where P_n denotes the Legendre polynomials. The polynomials P_n satisfy the following integral representation:

$$P_n(\cos u) = \frac{1}{\pi}\int_0^{\pi}(\cos u + i\sin u\cos\eta)^n\,d\eta. \tag{3.2}$$

Let us now suppose that expansion (3.1) converges to a function $\underline{f}(\cos u)$ but, for the moment, we shall leave the type of convergence unspecified. We only assume that $\underline{f}(\cos u)$ is a measurable and integrable function in the interval $u \in [0,\pi]$. Thus, the Legendre coefficients a_n can be written as

$$a_n = 2\pi\int_0^{\pi}\underline{f}(\cos u)P_n(\cos u)\sin u\,du. \tag{3.3}$$

Our goal now is to rewrite expansion (3.1) as a trigonometrical series. For this purpose, we prove the following proposition.

Proposition 2. *The Legendre coefficients $\{a_n\}_0^{\infty}$ coincide with the Fourier coefficients of the form:*

$$a_n = \int_{-\pi}^{\pi}\hat{f}(t)e^{int}\,dt \qquad (n=0,1,2,\ldots), \tag{3.4}$$

where

$$\hat{f}(t) = -2i\epsilon(t)e^{it/2}\int_0^t f(u)\,[2(\cos u-\cos t)]^{-1/2}\sin u\,du, \tag{3.5}$$

with $f(u)\equiv\underline{f}(\cos u)$, and $\epsilon(t)$ being the sign function.

Proof. From the Dirichlet–Murphy integral representation of the Legendre polynomials (see Ref. [9], Ch. III, Section 5.4):

$$P_n(\cos u) = -\frac{i}{\pi}\int_u^{(2\pi-u)} e^{i(n+1/2)t}\,[2(\cos u-\cos t)]^{-1/2}\,dt, \tag{3.6}$$

and from equality (3.3), we have

$$\frac{ia_n}{2} = \int_0^{\pi}du\,\underline{f}(\cos u)\sin u\int_u^{(2\pi-u)} e^{i(n+1/2)t}\,[2(\cos u-\cos t)]^{-1/2}\,dt. \tag{3.7}$$

Inverting the order of integration in formula (3.7), we get

$$\begin{aligned}\frac{ia_n}{2} = &\int_0^{\pi}dt\,e^{i(n+1/2)t}\int_0^t du\,\sin u\,\underline{f}(\cos u)\,[2(\cos u-\cos t)]^{-1/2}\\ &+\int_{\pi}^{2\pi}dt\,e^{i(n+1/2)t}\int_0^{(2\pi-t)}du\,\sin u\,\underline{f}(\cos u)\,[2(\cos u-\cos t)]^{-1/2}.\end{aligned} \tag{3.8}$$

Next, from the second integral in the r.h.s. of formula (3.8), if we perform the following change of variable, $t \to t - 2\pi$, and change $u \to -u$, we get

$$e^{i\pi} \int_{-\pi}^{0} dt\, e^{i(n+1/2)t} \int_{0}^{t} du\, \sin u\, \underline{f}(\cos u)\, [2(\cos u - \cos t)]^{-1/2}\,.$$

Finally, we obtain

$$\begin{aligned} \frac{ia_n}{2} = &\int_{0}^{\pi} dt\, e^{i(n+1/2)t} \int_{0}^{t} du\, \sin u\, \underline{f}(\cos u)\, [2(\cos u - \cos t)]^{-1/2} \\ &+ e^{i\pi} \int_{-\pi}^{0} dt\, e^{i(n+1/2)t} \int_{0}^{t} du\, \sin u\, \underline{f}(\cos u)\, [2(\cos u - \cos t)]^{-1/2}\,, \end{aligned} \tag{3.9}$$

which gives

$$a_n = \int_{-\pi}^{\pi} \hat{f}(t)\, e^{int}\, dt, \tag{3.10}$$

with $\hat{f}(t)$ given by (3.5).
(For a proof of this result in a more general setting see also [4]III). □

It can easily be verified that (see formula (3.5))

$$\hat{f}(t) = -e^{it} \hat{f}(-t), \tag{3.11}$$

and, accordingly, from (3.10) and (3.11) we have

$$a_n = -a_{-n-1} \qquad (n \in \mathbb{Z}). \tag{3.12}$$

We are thus prompted to consider the following trigonometrical series,

$$\begin{aligned} \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} a_n\, e^{-int} &= \frac{1}{2\pi} \left\{ \sum_{n=0}^{+\infty} a_n\, e^{-int} - e^{it} \sum_{n=0}^{+\infty} a_n\, e^{int} \right\} \\ &= \frac{1}{2\pi}\, e^{i\frac{(t-\pi)}{2}} \sum_{n=-\infty}^{+\infty} (-1)^n a_n\, \cos\left[\left(n+\frac{1}{2}\right)(t-\pi)\right] \\ &= \frac{1}{2\pi}\, e^{i\frac{(t-\pi)}{2}} \sum_{n=-\infty}^{+\infty} a_n\, \sin\left[\left(n+\frac{1}{2}\right) t\right], \end{aligned} \tag{3.13}$$

and study the holomorphic extension associated with it.

4. Holomorphic extension associated with the trigonometrical series

In the complex plane $\mathbb{C}$ of the variable $\tau = t + iw$ $(t, w \in \mathbb{R})$ we introduce the following domains: $\mathcal{I}_+^{(\pm\xi_0)} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > \pm\xi_0,\, \xi_0 \geq 0\}$, and $\mathcal{I}_-^{(\pm\xi_0)} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau < \pm\xi_0,\, \xi_0 \geq 0\}$. Correspondingly, we introduce the following cut–domains: $\mathcal{I}_+^{(\xi_0)} \setminus \Xi_+^{(\xi_0)}$, where $\Xi_+^{(\xi_0)} = \{\tau \in \mathbb{C} \mid \tau = 2k\pi + iw,\, w > \xi_0,\, \xi_0 \geq 0,\, k \in \mathbb{Z}\}$, and $\mathcal{I}_-^{(\xi_0)} \setminus \Xi_-^{(-\xi_0)}$, where $\Xi_-^{(-\xi_0)} = \{\tau \in \mathbb{C} \mid \tau = 2k\pi + iw,\, w < -\xi_0,\, \xi_0 \geq 0,\, k \in \mathbb{Z}\}$. We shall use the notation $\dot{A} = A \setminus 2\pi\mathbb{Z}$ for every subset A of $\mathbb{C}$ which is invariant under the translation group $2\pi\mathbb{Z}$. We can then prove the following proposition.

Proposition 3. *Let us consider the following trigonometrical series,*

$$\frac{1}{2\pi} \sum_{n=0}^{\infty} a_n\, e^{-in\tau} \qquad (\tau = t + iw;\, t, w \in \mathbb{R}), \tag{4.1}$$

and suppose that the set of numbers $\{f_n\}_0^\infty$*,* $f_n = n^p a_n$*,* $(n = 0, 1, 2, \ldots, p \geq 1)$ *satisfies condition (2.3). Then:*

i) *The series (4.1) converges to a function* $\hat{f}(\tau)$ *holomorphic in* $\mathcal{I}_-^{(0)}$*, the convergence being uniform in any compact subdomain of* $\mathcal{I}_-^{(0)}$*.*

ii) *The function* $\hat{f}(\tau)$ *admits a holomorphic extension to the cut–domain* $\mathcal{I}_+^{(0)} \setminus \dot{\Xi}_+^{(0)}$*, i.e., it is analytic in* $\mathbb{C} \setminus \{\tau = 2k\pi + iw \,|\, k \in \mathbb{Z}, w > 0\}$*.*

iii) *The jump function* $\hat{F}(w)$ *(which equals the discontinuity of* $i\hat{f}(\tau)$ *across the cuts* $\dot{\Xi}_+^{(0)}$*) is a function of class* C^{p-1}*,* $(p \geq 1)$*, and satisfies the following bound*

$$\left|\hat{F}(w)\right| \leq \|\tilde{a}_\sigma\|_1 \; e^{\sigma w}, \qquad \left(\sigma \geq -\frac{1}{2}, w \in \mathbb{R}^+\right), \tag{4.2}$$

where $\tilde{a}(\sigma + i\nu)$ $(\nu \in \mathbb{R})$ *is the Carlsonian interpolation of the coefficients* a_n*, and*

$$\|\tilde{a}_\sigma\|_1 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\tilde{a}(\sigma + i\nu)| \, d\nu, \qquad \left(\sigma \geq -\frac{1}{2}\right). \tag{4.3}$$

iv) $\tilde{a}(\sigma + i\nu)$ *is the Laplace transform of the jump function* $\hat{F}(w)$*: i.e.,*

$$\tilde{a}(\sigma + i\nu) = \int_0^{+\infty} \hat{F}(w) \, e^{-(\sigma + i\nu)w} \, dw, \qquad \left(\sigma > -\frac{1}{2}\right). \tag{4.4}$$

v) *The following Plancherel equality holds true:*

$$\int_{-\infty}^{+\infty} |\tilde{a}(\sigma + i\nu)|^2 \, d\nu = 2\pi \int_{-\infty}^{+\infty} \left|\hat{F}(w) \, e^{-\sigma w}\right|^2 dw, \qquad \left(\sigma \geq -\frac{1}{2}\right). \tag{4.5}$$

Proof. See the proof of Theorem 1 in [5] (see also the remark at the end of Proposition 1). □

Remark. Let us note that the Plancherel equality (4.5) holds true under the milder condition that the coefficients a_n $(n = 0, 1, 2, \ldots)$ satisfy condition (2.3).

Next, we can state the following proposition.

Proposition 4. *If in the trigonometrical series*

$$\frac{1}{2\pi} \left\{ \sum_{n=0}^{\infty} a_n \, e^{-int} - e^{it} \sum_{n=0}^{\infty} a_n \, e^{int} \right\} \qquad (t \in \mathbb{R}), \tag{4.6}$$

the coefficients a_n *satisfy the assumptions required by Proposition 3, then:*

i) *the series converges to a continuous function* $\hat{f}(t)$*,* $(t \in \mathbb{R})$*, the convergence being uniform on any compact subdomain of the real line.*

ii) *The function* $\hat{f}(t)$ *admits a holomorphic extension to the cut–domain* $\mathcal{I}_+^{(0)} \setminus \dot{\Xi}_+^{(0)} \cup \mathcal{I}_-^{(0)} \setminus \dot{\Xi}_-^{(0)}$*: i.e., it is analytic in* $\mathbb{C} \setminus \{\tau = 2k\pi + iw \,|\, k \in \mathbb{Z}, |w| > 0\}$*.*

iii) *The jump function across the cuts* $\dot{\Xi}_\pm^{(0)}$ *satisfies conditions analogous to (iii)–(v) of Proposition 3.*

Proof. Statement (i) follows from observing that, in view of the assumptions on the coefficients a_n we have:

$$\left| \frac{1}{2\pi} \left\{ \sum_{n=0}^{\infty} a_n \, e^{-int} - e^{it} \sum_{n=0}^{\infty} a_n \, e^{int} \right\} \right| \leq \frac{1}{\pi} \sum_{n=0}^{\infty} |a_n| < \infty. \tag{4.7}$$

Next, by the Weierstrass theorem on the uniformly convergent series of continuous functions, we obtain the result. Statements (ii) and (iii) can be proved analogously to the proof of the corresponding statements in Proposition 3. $\square$

Remarks. (i) If the coefficients a_n in (4.6) are exponentially bounded, i.e., $|a_n| \leq K\exp(-(n-m)\xi_0)$, $(n > m,\, m \in \mathbb{R}^+,\, \xi_0 > 0,\, K = \text{constant})$, then $\hat{f}(t)$ admits a holomorphic extension to the cut–domain $\mathfrak{I}_+^{(0)} \setminus \dot{\Xi}_+^{(\xi_0)} \cup \mathfrak{I}_-^{(0)} \setminus \dot{\Xi}_-^{(-\xi_0)}$ (see Proposition 5 of [5]). For the sake of simplicity, in the following we shall only consider the case $\xi_0 = 0$.
(ii) Similarly, we assume hereafter that condition (2.3) is satisfied by the whole sequence $\{f_n\}_0^\infty$; we could also assume that this condition is satisfied only by the subset $\{f_n\}_{n_0}^\infty$, $(n_0 > 0)$, $f_n = n^p a_n$, $p \geq 1$. In this case the result proved above still holds but for minor modifications, e.g., that in (4.2) now $\sigma \geq (n_0 - \frac{1}{2})$, $(n_0 > 0)$, and likewise in formulae (4.3), (4.4), and (4.5). See also the remark after Proposition 5 in [5].

5. Inversion of the Radon–Abel transformation and holomorphic extension associated with the Legendre Series

Proposition 2 allows us to regard the Legendre expansions as trigonometrical series. The function $\hat{f}(t)$ is the Radon–Abel transformation of the function $f(u)$ and, moreover, it can be regarded as the restriction of a function $\hat{f}(\tau)$, $(\tau \in \mathbb{C})$, which is the Radon–Abel transformation of a function $f(\theta)$, $(\theta \in \mathbb{C})$, when $\tau = t$ and $\theta = u$ $(t, u \in \mathbb{R})$ (see the Appendix). It is therefore of primary interest to derive the inversion of the Radon–Abel transformation. We can prove the following proposition.

Proposition 5. *Let us suppose that the sequence $f_n = n^p a_n$, $(p \geq 2)$, (the numbers a_n being the coefficients of the Legendre expansion (3.1)) satisfies the Hausdorff condition (2.3). We can then write the following Radon–Abel transformation (see formula (A.20), and (A.21) of the Appendix),*

$$\hat{f}(t) = -2e^{it/2}\int_0^t f(u)\,[2(\cos t - \cos u)]^{-1/2}\sin u\,du, \tag{5.1}$$

which admits the following inversion:

$$f(u) = \frac{1}{\pi\sin u}\frac{d}{du}\int_0^u e^{-it/2}\hat{f}(t)\,[2(\cos u - \cos t)]^{-1/2}\sin t\,dt. \tag{5.2}$$

Proof. In view of the assumptions on the Legendre coefficients a_n and of Propositions 2, 3 and 4 we can guarantee that representation (5.1) holds true, and, moreover, that the function $\hat{f}(t)$, $(t \in \mathbb{R})$, is continuous. Furthermore, since the sequence $n^p a_n$ satisfies the Hausdorff condition (2.3) with $p \geq 2$, $\hat{f}(t)$ is also differentiable. This fact can easily be proved by differentiating the series at the l.h.s. of formula (3.13) term by term, and observing that it is majorized by the convergent series: const. $\sum_{n=0}^\infty |na_n| < \infty$.
Let us note that, for the sake of simplicity, we work with representation (5.1) instead of (3.5) (see also the Appendix, formulae (A.20) and (A.21)).

Next, we set in formula (5.1) $\cos t = (1-\rho)$, $(\rho > 0)$, $\cos u = (1-\rho')$, $(0 \leq \rho' \leq \rho)$. Thus formula (5.1) can be rewritten as

$$\hat{f}(t) = 2ie^{it/2} \int_0^\rho \underline{f}(1-\rho') \left[2(\rho-\rho')\right]^{-1/2} \, d\rho'. \tag{5.3}$$

Then we introduce the Riemann–Liouville integral $[I_\alpha \phi]$, which can be written as

$$[I_\alpha \phi](\rho) = \frac{1}{\Gamma(\alpha)} \int_0^\rho \phi(\rho')(\rho-\rho')^{(\alpha-1)} \, d\rho' \qquad (\alpha > 0), \tag{5.4}$$

and we have

$$\hat{f}(t) = i\sqrt{2\pi} e^{it/2} [I_{1/2}\phi](t), \tag{5.5}$$

where

$$\phi(\rho') = \underline{f}(1-\rho'). \tag{5.6}$$

If $[I_\alpha \phi]$ is α–times differentiable, then the following properties of the Riemann–Liouville integral can be applied:

$$\text{i)} \qquad I_\alpha \circ I_\beta = I_{\alpha+\beta} \quad (\alpha, \beta > 0) \quad (\text{in particular, } I_{1/2} \circ I_{1/2} = I_1). \tag{5.7}$$

$$\text{ii)} \qquad \left(\frac{d}{d\rho}\right)^\alpha [I_\alpha \phi](\rho) = \phi(\rho) \qquad (\alpha = 1, 2, \ldots). \tag{5.8}$$

Since $\hat{f}(t)$ is differentiable we can write, by using properties (i) and (ii):

$$\frac{d}{d\rho'} \left[I_{1/2}[I_{1/2}\phi]\right] = \frac{1}{\sqrt{\pi}} \frac{d}{d\rho'} \int_0^{\rho'} [I_{1/2}\phi](\rho)(\rho'-\rho)^{-1/2} d\rho = \phi(\rho'). \tag{5.9}$$

By applying the last equalities to our case, and in view of (5.5), we obtain formula (5.2). □

For what concerns the inversion of the Radon–Abel transformation in a more general setting see Ref. [1].

Proposition 4 proves that, if the numbers $n^p a_n$ $(p \geq 1)$ satisfy the Hausdorff condition (2.3), then $\hat{f}(t)$ is the restriction to the real axis of a function $\hat{f}(\tau)$ holomorphic in $\dot{\mathfrak{I}} = \left(\mathfrak{I}_+^{(0)} \setminus \dot{\Xi}_+^{(0)}\right) \cup \left(\mathfrak{I}_-^{(0)} \setminus \dot{\Xi}_-^{(0)}\right)$. Assuming hereafter that the Hausdorff condition (2.3) is satisfied by the sequence $f_n = n^p a_n$ with $p \geq 2$, we can extend representation (5.2) uniquely in the following way:

$$f(\theta) = \frac{1}{\pi \sin\theta} \frac{d}{d\theta} \int_{\gamma_\theta} e^{-i\tau/2} \hat{f}(\tau) \frac{\sin\tau}{[2(\cos\theta - \cos\tau)]^{1/2}} \, d\tau, \tag{5.10}$$

where γ_θ denotes the ray $\underline{\gamma}_\theta$ oriented from 0 to θ. We can now prove the following proposition.

Proposition 6. *Let us suppose that the sequence $f_n = n^p a_n$ $(n = 0, 1, \ldots)$ satisfies the Hausdorff condition (2.3) with $p \geq 2$, then the function $f(\theta)$ represented by formula (5.10) is even, 2π–periodic, and holomorphic in $\dot{\mathfrak{I}}$ (here referred to the complex plane of the variable $\theta = u + iv$).*

Proof. The assumptions on the Legendre coefficients a_n allow us to state that $\hat{f}(\tau)$ is a 2π–periodic function holomorphic in the domain $\dot{\mathfrak{I}}$ of the complex τ–plane (see Proposition 4) that also satisfies the following symmetry property:

$$\hat{f}(\tau) = -e^{i\tau} \hat{f}(-\tau), \tag{5.11}$$

which derives from equality (3.11) in view of the uniqueness of the analytic continuation (see also the Appendix, formula (A.17)). The above–mentioned properties imply that $\hat{f}(\tau)$ is of the following form: $\hat{f}(\tau) = \exp(i\tau/2)(1-\cos\tau)^{1/2}\, b(\cos\tau)$, with $b(\cos\tau)$ analytic in $\underline{D} = \{\cos\tau \in \mathbb{C}, \tau \in \dot{\mathfrak{I}}\}$. Through the following parametrization of γ_θ: $\cos\tau = 1 + \lambda(\cos\theta - 1)$, $(0 \leq \lambda \leq 1)$ (see also the Appendix), the r.h.s. of formula (5.10) can be rewritten as

$$\frac{i}{\sqrt{2}\pi}\frac{d}{d(\cos\theta)}\left\{(\cos\theta-1)\int_0^1 b[1+\lambda(\cos\theta-1)]\lambda^{1/2}(1-\lambda)^{-1/2}d\lambda\right\}, \tag{5.12}$$

which represents an even, 2π–periodic function, holomorphic in the domain $\dot{\mathfrak{I}}$ of the complex θ–plane. Since in the following we shall prove that this function can be represented by the Legendre expansion (3.1), then it can properly be denoted by $f(\theta)$. □

Formula (5.10) allows us to compute the boundary values $f_\pm(v)$ (defined by $f_\epsilon(v) = \lim_{u\to 0^+} f(\epsilon u + iv)$; $\epsilon = \pm$; $v \geq 0$) on the semiaxis $\{\theta = iv,\ v \geq 0\}$ in terms of the corresponding boundary values $\hat{f}_\pm(w)$ (with $\gamma_{iv} : \{\tau = iw,\ 0 \leq w \leq v\}$), provided $\hat{f}_\pm(w)$ satisfy a C^1–type regularity condition; the latter is definitely necessary in order to perform the inversion of the Radon–Abel transform at the boundary. The C^1–continuity of the boundary values follows from the fact that the sequence $f_n = n^p a_n$, $p \geq 2$, satisfies the Hausdorff condition (2.3). We thus obtain:

$$i[f_+(v) - f_-(v)] = F(v) = \frac{1}{\pi\sinh v}\frac{d}{dv}\int_0^v e^{w/2}\,\hat{F}\,(w)\frac{\sinh w}{[2(\cosh v-\cosh w)]^{1/2}}\,dw, \tag{5.13}$$

$(\hat{F}(w) = i[\hat{f}_+(w) - \hat{f}_-(w)]$; $\hat{f}_\epsilon(w) = \lim_{t\to 0^+}\hat{f}(\epsilon t + iw)$; $\epsilon = \pm)$.
We can then apply the inverse Radon–Abel transform operator (defined by formula (5.10)) to the series at the r.h.s. of formula (3.13): i.e.,

$$\hat{f}(t) = \frac{1}{2\pi}e^{i(t-\pi)/2}\sum_{n=-\infty}^{+\infty}(-1)^n\,a_n\,\cos\left[\left(n+\frac{1}{2}\right)(t-\pi)\right], \tag{5.14}$$

and integrate term by term in view of the uniform convergence of this series, which follows from the Hausdorff conditions on the coefficients a_n. Next, we introduce the functions

$$\begin{aligned}\psi_n(\cos u) = -\frac{i}{\pi\sin u}\frac{d}{du}\int_0^u \cos\left[\left(n+\frac{1}{2}\right)(t-\pi)\right]&\\ \times\frac{\sin t}{[2(\cos u-\cos t)]^{1/2}}\,dt \qquad (0<u<2\pi),&\end{aligned} \tag{5.15}$$

which are related to the Legendre polynomials $P_n(\cos u)$ as follows (see formulae (II.79) and (II.91) of [4]II):

$$\psi_n(\cos u) = \frac{(-1)^n}{4}(2n+1)P_n(\cos u). \tag{5.16}$$

Finally, recalling that $a_n = -a_{-n-1}$ $(n \in \mathbb{Z})$, we obtain again the original Legendre expansion (3.1):

$$f(u) = \underline{f}(\cos u) = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n a_n\,\psi_n(\cos u) = \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)a_n\,P_n(\cos u). \tag{5.17}$$

We can restate the results of Proposition 6 in the more natural geometry of the $\cos\theta$–plane.

Proposition 7. *If the sequence $f_n = n^p a_n$ $(n = 0, 1, \ldots)$ satisfies the Hausdorff condition (2.3) with $p \geq 2$, then:*

i) *the series (1.1) converges uniformly to an analytic function $\underline{f}(\cos u)$ $(u \equiv \mathrm{Re}\,\theta)$ in any compact domain $|\cos u| \leq |\cos u_0| < 1$; and*
ii) *the function $\underline{f}(\cos u)$ admits a holomorphic extension to the complex $\cos\theta$–plane $(\theta = u + iv)$ cut along the axis $[1, +\infty)$.*

6. Laplace transformation, representation of the jump function and its reconstruction by the use of the Pollaczek polynomials

From formulae (4.4) and (A.7) the following equality follows:

$$\tilde{a}(\lambda) = \int_0^{+\infty} e^{-(\lambda+1/2)w} (\mathcal{A}F)(w)\,dw \qquad \left(\lambda = \sigma + i\nu,\ \mathrm{Re}\,\lambda > -\frac{1}{2}\right). \tag{6.1}$$

Writing explicitly the Abel transform $(\mathcal{A}F)(w)$ (see formula (A.7)) yields

$$\tilde{a}(\lambda) = 2\int_0^{+\infty} e^{-(\lambda+1/2)w} \left\{\int_0^w \frac{\underline{F}(\cosh v)\sinh v}{[2(\cosh w - \cosh v)]^{1/2}}\,dv\right\} dw \qquad \left(\mathrm{Re}\,\lambda > -\frac{1}{2}\right). \tag{6.2}$$

If we exchange the integration order, this becomes

$$\tilde{a}(\lambda) = 2\int_0^{+\infty} \underline{F}(\cosh v)\sinh v \left\{\int_v^{+\infty} \frac{e^{-(\lambda+1/2)w}}{[2(\cosh w - \cosh v)]^{1/2}}\,dw\right\} dv \qquad \left(\mathrm{Re}\,\lambda > -\frac{1}{2}\right). \tag{6.3}$$

Recalling the integral representation of the second–kind Legendre functions $Q_\lambda(\cosh v)$, i.e.,

$$Q_\lambda(\cosh v) = \int_v^{+\infty} \frac{e^{-(\lambda+1/2)w}}{[2(\cosh w - \cosh v)]^{1/2}}\,dw \qquad (\mathrm{Re}\,\lambda > -1, v > 0), \tag{6.4}$$

we can write formula (6.3) as follows

$$\tilde{a}(\lambda) = 2\int_0^{+\infty} \underline{F}(\cosh v) Q_\lambda(\cosh v)\sinh v\,dv \qquad \left(\mathrm{Re}\,\lambda > -\frac{1}{2}\right). \tag{6.5}$$

Remark. The second–kind Legendre function presents a logarithmic singularity at $v = 0$; then the integral representation (6.4) holds true if $v > 0$; nevertheless, the integral in (6.5) converges if $\underline{F}(\cosh v)$ is regular at $v = 0$.

If $\mathrm{Re}\,\lambda = -1/2$ we can split $Q_{-1/2+i\nu}(\cosh v)$ into two terms, $Q^{(E)}_{-1/2+i\nu}(\cosh v)$ and $Q^{(O)}_{-1/2+i\nu}(\cosh v)$, defined as follows:

$$Q^{(E)}_{-1/2+i\nu}(\cosh v) = \int_v^{+\infty} \frac{\cos\nu w}{[2(\cosh w - \cosh v)]^{1/2}}\,dw \qquad (v > 0), \tag{6.6}$$

$$Q^{(O)}_{-1/2+i\nu}(\cosh v) = -i\int_v^{+\infty} \frac{\sin\nu w}{[2(\cosh w - \cosh v)]^{1/2}}\,dw \qquad (v > 0). \tag{6.7}$$

Next, we recall the following equality (see [2]):

$$P_\lambda(\cos\theta) = \tan(\pi\lambda)\{Q_\lambda(\cos\theta) - Q_{-\lambda-1}(\cos\theta)\}, \tag{6.8}$$

(where we use a non–standard normalization of the Q_λ functions, which is more appropriate to our joint consideration of P_λ and Q_λ; the discrepancy with the usual notation is a factor $1/\pi$). We thus have the following equality, which will be useful later on:

$$P_{-1/2+i\nu}(\cosh v) = P_{-1/2-i\nu}(\cosh v) = 2\tan\left[\pi\left(-\frac{1}{2}+i\nu\right)\right] Q^{(O)}_{-1/2+i\nu}(\cosh v). \tag{6.9}$$

We can now prove the following proposition.

Proposition 8. *If the sequence $f_n = n^p a_n$ (a_n being the Legendre coefficients) satisfies the Hausdorff condition (2.3) with $p \geq 2$, then the jump function $F(v) = \underline{F}(\cosh v)$ admits the integral representation*

$$F(v) = \underline{F}(\cosh v) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} \tilde{a}(\sigma+i\nu)h(\sigma+i\nu)P_{\sigma+i\nu}(\cosh v)\,d\nu \qquad \left(\sigma \geq -\frac{1}{2}\right), \tag{6.10}$$

where $h(\sigma+i\nu) = 2(\sigma+i\nu)+1$, and $P_{\sigma+i\nu}(\cosh v)$ denotes the first–kind Legendre functions.

Proof. In Propositions 3 and 4 we derived the following formula:

$$\hat{F}(w) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{a}(\sigma+i\nu)\,e^{(\sigma+i\nu)w}\,d\nu \qquad \left(\sigma \geq -\frac{1}{2}\right). \tag{6.11}$$

This formula can indeed be obtained by evaluating the discontinuity across $\tau = iw$, $w \geq 0$ of the function $g(\tau) - \exp(i\tau)g(-\tau)$, where $g(\tau) = \frac{1}{2\pi}\sum_{n=0}^{\infty} a_n \exp(-in\tau)$. Note, in fact, that the jump of $g(-\tau)$ across this half–line is equal to zero; but its expression through a vanishing Cauchy integral allows us to introduce another equivalent integral representation of $\hat{F}(w)$, i.e.,

$$\hat{F}(w) = \frac{i}{\pi}e^{-w/2}\int_{-\infty}^{+\infty} \tilde{a}(\sigma+i\nu)\sin\left\{\left[\nu - i\left(\sigma+\frac{1}{2}\right)\right]w\right\}\,d\nu \qquad \left(\sigma \geq -\frac{1}{2}\right). \tag{6.12}$$

Applying the inverse Radon–Abel transform to $\exp(w/2)\hat{F}(w)$ (see formula (5.13)) yields

$$F(v) = -\frac{i}{\pi^2}\int_{-\infty}^{+\infty} \tilde{a}(\sigma+i\nu)\left\{\frac{1}{\sinh v}\frac{d}{dv}\int_0^v \frac{\sin\left\{\left[i\left(\sigma+\frac{1}{2}\right)-\nu\right]w\right\}\sinh w}{\left[2(\cosh v - \cosh w)\right]^{1/2}}\,dw\right\}d\nu$$
$$\left(\sigma \geq -\frac{1}{2}\right). \tag{6.13}$$

The r.h.s. of formula (6.13) converges to $F(v)$ if $\hat{F}(w)$ is of class C^1 and $\nu\tilde{a}(\sigma+i\nu)$ belongs to $L^1(-\infty,+\infty)$. Both properties follow from the requirement that the sequence $f_n = n^p a_n$ satisfies the Hausdorff condition (2.3) with $p \geq 2$. Finally we recognize in the integrand of formula (6.13) the first–kind Legendre function $P_{\sigma+i\nu}(\cosh v)$; in fact, we have (see formula (II.86) of [4]II):

$$\begin{aligned}\frac{1}{4\pi}&P_{\sigma+i\nu}(\cosh v)[2(\sigma+i\nu)+1] \\ &= -\frac{i}{\pi^2\sinh v}\frac{d}{dv}\int_0^v \frac{\sin\left\{\left[i\left(\sigma+\frac{1}{2}\right)-\nu\right]w\right\}}{\left[2(\cosh v - \cosh w)\right]^{1/2}}\sinh w\,dw.\end{aligned} \tag{6.14}$$

By plugging (6.14) into (6.13), we get the result, i.e., formula (6.10). $\square$

In the particular case of $\sigma = -1/2$, in view of the evenness of the function $P_{-1/2+i\nu}(\cosh v)$ with respect to ν, only the odd component of $\tilde{a}(-1/2+i\nu)$ contributes to the integral in (6.10). Accordingly, in view of formula (6.9), we can write the Laplace transform (6.5) in terms of the function $P_{-1/2+i\nu}(\cosh v)/\tan[\pi(-1/2+i\nu)]$, instead of $Q_{-1/2+i\nu}(\cosh v)$. It can easily be verified that, in this case, formulae (6.5) and (6.10) give (up to normalization constants) the classical Mehler transform (see [2]), which is precisely the tool used by Stein and Wainger [8] for proving their theorem.

We can now rapidly mention how the discontinuity function can be reconstructed, starting from the Fourier coefficients, by the use of the Pollaczek polynomials.

Proposition 9. *Let us suppose that the sequence of the Fourier–Legendre coefficients a_n, $(n = 0, 1, 2, \ldots)$ satisfies the Hausdorff condition (2.3); then the function $\hat{F}(w)e^{w/2}$ (see formula (4.5)), can be represented by the following expansion which converges in the sense of the L^2–norm:*

$$\hat{F}(w)e^{w/2} = \sum_{\ell=0}^{\infty} c_\ell \Phi_\ell(w) \qquad (w \in \mathbb{R}^+), \tag{6.15}$$

where

$$c_\ell = \sqrt{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a_n \mathcal{P}_\ell\left[-i\left(n+\frac{1}{2}\right)\right], \tag{6.16}$$

$$\Phi_\ell(w) = i^\ell \sqrt{2} e^{-w/2} L_\ell(2e^{-w}) e^{-e^{-w}}, \tag{6.17}$$

where $\mathcal{P}_\ell$ and L_ℓ are the Pollaczek and the Laguerre polynomials [2], *respectively.*

Proof. Let us note that $\tilde{a}(-1/2+i\nu)$ and $\hat{F}(w)e^{w/2}$ belong to $L^2(-\infty,+\infty)$ (see statement (iii) of Proposition 1 and formula (4.5)). We can then write

$$\hat{F}(w)e^{w/2} = \underset{\nu_0\to+\infty}{\mathrm{l.i.m.}} \left(\frac{1}{2\pi}\int_{-\nu_0}^{\nu_0} \tilde{a}\left(-\frac{1}{2}+i\nu\right) e^{i\nu w}\, d\nu\right), \tag{6.18}$$

and the proof proceeds exactly as in Theorem 2 of [5], where $\hat{F}(w)e^{w/2}$ now plays the role of $F(v)e^{v/2}$ (see also the Remark at the end of Proposition 1). $\square$

Remark. The previous result holds true under the condition that the sequence $f_n = n^p a_n$, $(n = 0, 1, \ldots)$ satisfies condition (2.3) with $p \geq 0$; but in the next proposition, which involves the Abel transform and its inverse, the latter condition must be satisfied with $p \geq 2$ (see formulae (5.13) and (A.7)).

Next we set (see formula (A.7)): $\cosh v = y$, $\cosh w = x$; we can rewrite the Abel transform as a convolution product of the following form

$$(\mathcal{A}F)(x) = \sqrt{2}\int_1^x \frac{\underline{F}(y)}{\sqrt{(x-y)}}\, dy :\equiv \sqrt{2}\,(\underline{F} * \chi_1)(x). \tag{6.19}$$

From formula (5.13) we analogously obtain

$$\underline{F}(y) = \frac{1}{\pi\sqrt{2}}\frac{d}{dy}\int_1^y \frac{(\mathcal{A}F)(x)}{\sqrt{(y-x)}}\, dx :\equiv \frac{1}{\pi\sqrt{2}}\frac{d}{dy}(\mathcal{A}F * \chi_1)(y). \tag{6.20}$$

We can then prove the following proposition.

Proposition 10. *If the sequence $f_n = n^p a_n$ $(n = 0, 1, 2, \ldots)$ satisfies the Hausdorff condition (2.3) with $p \geq 2$, then the following limit holds true:*

$$\lim_{m\to\infty} \langle(\underline{F} - \psi_m), \phi\rangle = 0, \tag{6.21}$$

where

$$\psi_m = \frac{1}{\pi\sqrt{2}} \frac{d}{dy}\left[\left(\sum_{\ell=0}^{m} c_\ell \Phi_\ell\right) * \chi_1\right], \tag{6.22}$$

$\phi \in S_\infty(\mathbb{R})$ ($S_\infty(\mathbb{R})$ being the Schwartz space of the $C^\infty(\mathbb{R})$ functions $\phi(x)$ that together with all their derivatives decrease, for $|x|$ tending to ∞, faster than any negative power of $|x|$), and $\langle f, \phi\rangle$ denotes the Lebesgue integral $\int_{-\infty}^{+\infty} \bar{f}\phi\,dx$.

Proof. We have from formulae (6.20) and (6.22):

$$\begin{aligned}\langle(\underline{F} - \psi_m), \phi\rangle &= \frac{1}{\pi\sqrt{2}}\left\langle \frac{d}{dy}\left[\left(\mathcal{A}F - \sum_{\ell=0}^{m} c_\ell\Phi_\ell\right) * \chi_1\right], \phi\right\rangle \\ &= -\frac{1}{\pi\sqrt{2}}\left\langle \left(\mathcal{A}F - \sum_{\ell=0}^{m} c_\ell\Phi_\ell\right) * \chi_1, \phi'\right\rangle \qquad (\phi \in S_\infty(\mathbb{R})).\end{aligned} \tag{6.23}$$

From inequality (4.2) and formula (A.7) it follows that $\mathcal{A}F$ has a power–like behavior in x. Next, in view of the Fubini theorem, we have:

$$\left\langle \left(\mathcal{A}F - \sum_{\ell=0}^{m} c_\ell\Phi_\ell\right) * \chi_1, \phi'\right\rangle = \left\langle \mathcal{A}F - \sum_{\ell=0}^{m} c_\ell\Phi_\ell, \phi' * \chi^\infty\right\rangle, \tag{6.24}$$

where

$$(\phi' * \chi^\infty)(x) = \int_x^{+\infty} \frac{\phi'(y)}{\sqrt{(y-x)}}\,dy < \infty. \tag{6.25}$$

From the Schwarz inequality it follows:

$$\left\langle \mathcal{A}F - \sum_{\ell=0}^{m} c_\ell\Phi_\ell, \phi' * \chi^\infty\right\rangle \leq \left\|\mathcal{A}F - \sum_{\ell=0}^{m} c_\ell\Phi_\ell\right\|_{L^2[0,+\infty)} \cdot \|\phi' * \chi^\infty\|_{L^2[0,+\infty)}. \tag{6.26}$$

Now, in view of the fact that $\lim_{m\to\infty} \|\mathcal{A}F - \sum_{\ell=0}^{m} c_\ell\Phi_\ell\|_{L^2[0,+\infty)} = 0$ (see Proposition 9) and that $\|\phi' * \chi^\infty\|_{L^2[0,+\infty)} < \infty$, statement (6.21) holds true. □

Appendix A. Horocycles and Radon–Abel transformations on the (real and complexified) one–sheeted hyperboloids

Let X denote the real one–sheteed hyperboloid in $\mathbb{R}^3$, with equation:

$$x_0^2 - x_1^2 - x_2^2 = -1 \qquad (x \equiv (x_0, x_1, x_2)), \tag{A.1}$$

and let $\hat{X}$ be the meridian hyperbola, in the $x_1 = 0$ plane, with equation $x_0^2 - x_2^2 = -1$ (see Fig. 1). The manifold X can be described by the following polar coordinates:

$$x_0 = \sinh v \cosh\phi \qquad (v, \phi \in \mathbb{R}), \tag{A.2a}$$

$$x_1 = \sinh v \sinh\phi, \tag{A.2b}$$

$$x_2 = \cosh v. \tag{A.2c}$$

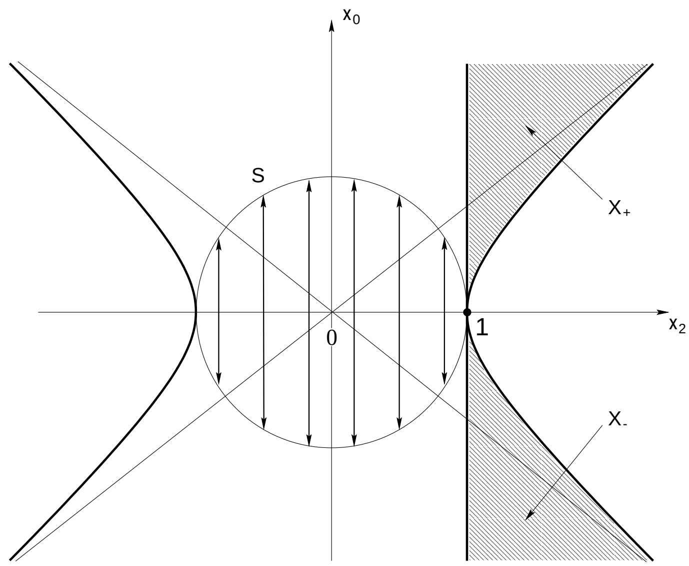

Figure 2: $x_1 = 0$ section of the one–sheteed hyperboloid.

Hereafter we shall deal more specifically with the following region of X (see Fig. 2):

$$X_+ = \{x \in X \,:\, x_0 \geq 0,\, x_2 \geq 1\}. \tag{A.3}$$

Besides polar coordinates, another system of local coordinates on X is equally valid for describing the set X_+, namely the horocyclic coordinates:

$$x_0 = \sinh w + \frac{1}{2}\zeta^2\, e^w \qquad (\zeta \in \mathbb{R},\, w \in \mathbb{R}^+), \tag{A.4a}$$

$$x_1 = \zeta\, e^w, \tag{A.4b}$$

$$x_2 = \cosh w - \frac{1}{2}\zeta^2\, e^w. \tag{A.4c}$$

The sections $w =$ const. are parabolae lying in the planes $x_0 + x_2 = e^w$, called horocycles.

Next we introduce the following integral:

$$\int_{h_w} \underline{F}\left(\cosh w - \frac{1}{2}\zeta^2\, e^w\right)\, d\zeta = \hat{F}(w), \tag{A.5}$$

where h_w is the oriented segment of horocycle belonging to X_+, which is represented by the arc of the parabola whose apex, which lies on $\hat{X}$, is obtained by setting $\zeta = 0$ in eqs. (A.4) (i.e., with coordinates $x_0 = \sinh w$, $x_1 = 0$, $x_2 = \cosh w$), and whose endpoints lie on the plane $x_2 = 1$. Moreover, the function $\underline{F}$ is assumed to satisfy the regularity conditions that make the integral (A.5) convergent. Since the integrand is an even function of ζ, the integration domain can be restricted to the part of h_w with $x_1 \geq 0$, which is given by

$$h_w^+ : \left\{\zeta = \left[2e^{-w}\,(1-\lambda)(\cosh w - 1)\right]^{1/2},\, (0 \leq \lambda \leq 1),\, w \in \mathbb{R}^+\right\}. \tag{A.6}$$

For $\lambda = 1$ we get $\zeta = 0$, i.e., the apex of the parabola representing the horocycle; for $\lambda = 0$ we get $\zeta = \left[2e^{-w}(\cosh w - 1)\right]^{1/2}$, which gives the intersection of the horocycle with the plane $x_2 = 1$.

Integral (A.5) defines a transformation of Radon–type in X, where the horocycles play the same role as the planes do in the ordinary Radon transformation. Next, if we set $x_2 = \cosh v$, we have $\zeta(v) = \left[2e^{-w}(\cosh w - \cosh v)\right]^{1/2}$, which varies again between $\zeta = 0$ for $v = w$ (apex of the parabola) and $\zeta = \left[2e^{-w}(\cosh w - 1)\right]^{1/2}$ for $v = 0$ (endpoint of the parabola: i.e., $x_2 = \cosh v = 1$). Since $d\zeta/dv = -e^{-w/2}\sinh v[2(\cosh w - \cosh v)]^{-1/2}$, integral (A.5) takes the form:

$$\begin{aligned} \hat{F}(w) &= 2e^{-w/2}\int_0^w \underline{F}(\cosh v)\frac{\sinh v}{[2(\cosh w - \cosh v)]^{1/2}}\,dv \\ &:\equiv e^{-w/2}(\mathcal{A}F)(w)\ ,\ (w \in \mathbb{R}^+), \end{aligned} \tag{A.7}$$

which is an Abel–type integral.

We notice that the set of horocycles $\{h_w;\, w \geq 0\}$ defines a fibration with basis $\hat{X}$ on the domain X. We denote by $\underline{h}$ the projection associated with this fibration; in particular, all the points of the domain X_+ are projected on the basis $\hat{X}$, i.e., $\forall x \in X_+, (x = (x_0, x_1, x_2)), \underline{h}(x) = x_w$ is the intersection of $\hat{X}$ with the unique horocycle h_w which contains x.

We can now regard the one–sheeted hyperboloid X as a real submanifold of a complex hyperboloid $X^{(c)}$ (see Fig. 1), whose equation is given by:

$$z_0^2 - z_1^2 - z_2^2 = -1 \qquad (z = (z_0, z_1, z_2),\, z_i \in \mathbb{C},\, i = 0, 1, 2). \tag{A.8}$$

Accordingly, we introduce the following complex–valued polar coordinates:

$$z_0 = -i\sin\theta\cosh\phi \qquad (\theta, \phi \in \mathbb{C}), \tag{A.9a}$$

$$z_1 = -i\sin\theta\sinh\phi, \tag{A.9b}$$

$$z_2 = \cos\theta. \tag{A.9c}$$

If we set $\theta = iv$ ($v \in \mathbb{R}$) in eqs. (A.9), and assume ϕ real, we obtain the polar coordinates (A.2) that describe the real hyperboloid X; if we set $\theta = u$ ($u \in \mathbb{R}$) and $\phi = i\eta$ ($\eta \in \mathbb{R}$), we obtain the *Euclidean* sphere:

$$z_0 = -i\sin u\cos\eta \qquad (u, \eta \in \mathbb{R}), \tag{A.10a}$$

$$z_1 = \sin u\sin\eta, \tag{A.10b}$$

$$z_2 = \cos u. \tag{A.10c}$$

We now want to extend the fibration with basis $\hat{X}$, introduced above, to the complex one–sheeted hyperboloid $X^{(c)}$. For this purpose, we consider the complex meridian hyperbola $\hat{X}^{(c)}$, lying in the $z_1 = 0$ plane, with equation: $z_0^2 - z_2^2 = -1$. The intersections of $\hat{X}^{(c)}$ with the family of planes P_τ, with equations $z_0 + z_2 = e^{-i\tau}$ ($\tau \in \mathbb{C}$, $\tau = t + iw$) are the points z_τ whose coordinates are $(z_\tau)_0 = -i\sin\tau$, $(z_\tau)_2 = \cos\tau$. The sections of $X^{(c)}$ by the planes P_τ are complex parabolae (except in the case $z_0 + z_2 = 0$) lying in the planes P_τ. These complex parabolae are the geometric realization of the complex horocycles, which will be denoted by h_τ ($\tau \in \mathbb{C}$). At this point, it is convenient to introduce the domain $X'^{(c)} = \{z \in X^{(c)};\, z_0 + z_2 \neq 0\}$ (dense in $X^{(c)}$), which can be parametrized by the horocyclic coordinates (ζ, τ) in

the following way:

$$z_0 = -i\sin\tau + \frac{1}{2}\zeta^2 e^{-i\tau} \qquad (\zeta, \tau \in \mathbb{C}), \tag{A.11a}$$

$$z_1 = \zeta e^{-i\tau}, \tag{A.11b}$$

$$z_2 = \cos\tau - \frac{1}{2}\zeta^2 e^{-i\tau}. \tag{A.11c}$$

We can now introduce the cut–domain $X^{(c)} \setminus \Sigma^{(c)}$, where $\Sigma^{(c)}$ is defined as follows:

$$\Sigma^{(c)} = \{z \in X^{(c)};\ z_2 \in [1, +\infty)\}. \tag{A.12}$$

The domain $\Sigma^{(c)} \cap X$ is composed by two sets, i.e. X_+ (defined by (A.3)) and X_- defined by (see Fig. 2):

$$X_- = \{x \in X;\ x_0 \leq 0,\ x_2 \geq 1\}. \tag{A.13}$$

The fibration produced by the horocycles h_w can now be extended through the use of complex horocycles h_τ, whose intersections with the meridian (complex) hyperbola $\hat{X}^{(c)}$ are the points z_τ with coordinates $(z_\tau)_0 = -i\sin\tau$, $(z_\tau)_2 = \cos\tau$. In the following we shall deal with functions that only depend on the coordinate $z_2 = \cos\theta$, i.e., $\underline{f}(\cos\theta)$ (or, alternatively, $f(\theta)$). Furthermore, we assume that these even and 2π–periodic functions $f(\theta)$ are holomorphic in the cut–domain $\dot{\mathfrak{I}} = \mathfrak{I}_+^{(0)} \setminus \dot{\Xi}_+^{(0)} \cup \mathfrak{I}_-^{(0)} \setminus \dot{\Xi}_-^{(0)}$, and continuous up to the boundaries of this domain (notice that the notations $\mathfrak{I}_\pm^{(0)}$ and $\dot{\Xi}_\pm^{(0)}$ have been introduced in connection with Proposition 3 where they referred to the complex plane of the variable τ). Moreover, we denote by $\underline{D} \in \mathbb{C}$ the domain of analyticity of $\underline{f}(\cos\theta)$ in the $\cos\theta$–plane. Next, we introduce the following integral:

$$2\int_{h_\tau^+} \underline{f}\left(\cos\tau - \frac{1}{2}\zeta^2\, e^{-i\tau}\right) d\zeta = \hat{f}(\tau), \tag{A.14}$$

h_τ^+ being the arc of complex horocycle defined by

$$h_\tau^+ : \left\{\zeta = \left[2\, e^{i\tau}(1-\lambda)(\cos\tau - 1)\right]^{1/2},\ (0 \leq \lambda \leq 1),\ \tau \in \mathbb{C}\right\}. \tag{A.15}$$

For $\lambda = 1$, we get $\zeta = 0$: i.e., the point z_τ (belonging $\hat{X}^{(c)}$); for $\lambda = 0$, we get $\zeta = [2\exp(i\tau)(\cos\tau - 1)]^{1/2}$, which gives the intersection of h_τ^+ with the plane $z_2 = 1$ (in particular, if we set $\tau = iw$, $w \in \mathbb{R}^+$, we obtain the expression $\zeta(w) = [2\exp(-w)(\cosh w - 1)]^{1/2}$, previously established in connection with formula (A.6)). Then, by setting $z_2 = \cos\theta$ ($\theta \in \mathbb{C}$, $\theta = u + iv$), we have: $\zeta(\theta) = [2e^{i\tau}(\cos\tau - \cos\theta)]^{1/2}$, and $d\zeta = e^{i\tau/2}[2(\cos\tau - \cos\theta)]^{-1/2}\sin\theta\, d\theta$. Furthermore, since on h_τ^+ we have: $\cos\theta(\lambda) - 1 = \lambda(\cos\tau - 1)$ $(0 \leq \lambda \leq 1)$, integral (A.14) can be rewritten in the following form

$$\hat{f}(\tau) = -2e^{i\tau/2}\int_{\gamma_\tau} f(\theta)\,[2(\cos\tau - \cos\theta)]^{-1/2}\sin\theta\, d\theta, \tag{A.16}$$

where γ_τ denotes the *ray* $\underline{\gamma}_\tau$ oriented from 0 to τ, and $\underline{\gamma}_\tau : \{\theta = \theta(\lambda);\ \cos\theta(\lambda) - 1 = \lambda(\cos\tau - 1),\ 0 \leq \lambda \leq 1,\ \theta(0) = 0,\ \theta(1) = \tau\}$. Moreover, the relevant branch of the function $[2(\cos\tau - \cos\theta)]^{-1/2}$ is specified by the condition that for $\tau = iw$, and $\theta = iv$ (with $w > v$), it takes the value $[2(\cosh w - \cosh v)]^{-1/2} \geq 0$. In fact, when $\tau = iw$ $(w > 0)$, the horocycle $h_\tau = h_{iw}$ is real and carried by the hyperboloid X. Moreover, in this case, transformation (A.16) can be applied to the boundary

values of f on the opposite sides of the cut (corresponding to the domain X_+), and, in particular, to the corresponding discontinuity function; in this way formula (A.7) is reobtained, and the function $\underline{F}$ now represents the jump function across the cut.

We can now show that if $f(\theta)$ is an even 2π–periodic function holomorphic in the cut–domain $\dot{\mathfrak{I}}$, then $\hat{f}(\tau)$ is a 2π–periodic function holomorphic in $\dot{\mathfrak{I}}$ that satisfies the following symmetry relation:

$$\hat{f}(\tau) = -e^{i\tau}\,\hat{f}(-\tau). \tag{A.17}$$

In order to prove this statement we rewrite expression (A.16) in the following form

$$\hat{f}(\tau) = e^{i\tau/2}\,[2(\cos\tau - 1)]^{1/2} \int_0^1 \underline{f}(1+\lambda(\cos\tau - 1))(1-\lambda)^{-1/2}\,d\lambda, \tag{A.18}$$

where the following parametrization $\cos\theta(\lambda) = 1 + \lambda(\cos\tau - 1)$ in integral (A.16) has been used. Since the integral in (A.18) is a function of $\cos\tau$ analytic in $\underline{D}$, $\hat{f}(\tau)$ can be written as

$$\hat{f}(\tau) = e^{i\tau/2}\left(\sin\frac{\tau}{2}\right)\,a(\cos\tau), \tag{A.19}$$

where $a(\cos\tau)$ is a function holomorphic in $\underline{D}$. From representation (A.19) one recovers that $\hat{f}(\tau)$ is 2π–periodic and, therefore, holomorphic in $\dot{\mathfrak{I}}$, and, in addition, the symmetry relation (A.17) is satisfied.

Finally, by restricting formulae (A.14) and (A.16) to the set of real values of the variables τ and θ, namely $\tau = t$, $\theta = u$, from (A.16) we obtain:

$$\hat{f}(t) = -2\,e^{it/2} \int_0^t f(u)\,[2(\cos t - \cos u)]^{-1/2} \sin u\,du. \tag{A.20}$$

By taking into account the relevant branch of the factor $[2(\cos t - \cos u)]^{-1/2}$, formula (A.20) can be written in the following more precise form (involving a positive bracket),

$$\hat{f}(t) = -2i\epsilon(t)\,e^{it/2} \int_0^t f(u)\,[2(\cos u - \cos t)]^{-1/2} \sin u\,du, \tag{A.21}$$

where $\epsilon(t)$ denotes the sign function. Note that formula (A.21) coincides with formula (3.5).

Acknowledgments. The authors feel deeply indebted to Prof. J. Bros for many illuminating discussions.

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(E. De Micheli) IBF – Consiglio Nazionale delle Ricerche, Via De Marini, 6 - 16149 Genova, Italy
E-mail address, E. De Micheli: demicheli@ge.cnr.it

(G. A. Viano) Dipartimento di Fisica - Università di Genova, Istituto Nazionale di Fisica Nucleare - sez. di Genova, Via Dodecaneso, 33 - 16146 Genova, Italy
E-mail address, G.A. Viano: viano@ge.infn.it